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# Fundamental groups of curves in positive characteristic

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**ABSTRACT.** We present some recent results (mainly of the author) concerning fundamental groups of curves over algebraically closed fields of positive characteristic.

## §1. Main question (cf. [H2,4][T3]).

In this §, we introduce our main question.

Throughout this article, we let  $k$  denote an algebraically closed field of characteristic  $p > 0$ ,  $X$  a smooth curve over  $k$ ,  $X^*$  the smooth compactification of  $X$  and  $\Sigma \stackrel{\text{def}}{=} X^* - X$ . We define non-negative integers  $g$  and  $n$  to be the genus of the proper, smooth curve  $X^*$  and the cardinality of the point set  $\Sigma$ , respectively. Note that  $X$  is hyperbolic (resp. affine, resp. projective) if and only if  $2 - 2g - n < 0$ , i.e.,  $(g, n) \neq (0, 0), (0, 1), (0, 2), (1, 0)$  (resp.  $n > 0$ , resp.  $n = 0$ ).

We denote by  $k(X)^\sim$  the maximal separable algebraic extension of  $k(X)$  in which the discrete valuation ring  $\mathcal{O}_{X,x}$  is unramified for all  $x \in X$ , and by  $k(X)^{\sim t}$  the maximal separable algebraic extension of  $k(X)$  in which  $\mathcal{O}_{X,x}$  is unramified for all  $x \in X$  and at most tamely ramified for all  $x \in \Sigma$ . Then, the fundamental group  $\pi_1(X)$  (resp. the tame fundamental group  $\pi_1^t(X)$ ) of  $X$  is nothing but the Galois group  $\text{Gal}(k(X)^\sim/k(X))$  (resp.  $\text{Gal}(k(X)^{\sim t}/k(X))$ ).

### Definition.

(i) For a (discrete) group  $\Gamma$ , we denote by  $\Gamma^\sim$  its profinite completion

$$\varprojlim_{N \triangleleft \Gamma, (\Gamma:N) < \infty} \Gamma/N.$$

(ii) For a profinite group  $G$ , we denote by  $G^{p'}$  its maximal pro-prime-to- $p$  quotient

$$\varprojlim_{N \triangleleft G \text{ closed, } p \nmid (G:N) < \infty} G/N.$$

(iii) For non-negative integers  $g$  and  $n$ , we denote by  $\Pi_{g,n}$  the topological fundamental group of a compact orientable surface of genus  $g$  with  $n$  points deleted. More concretely,

$$\Pi_{g,n} = \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_n \mid \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \dots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} \gamma_1 \dots \gamma_n = 1 \rangle$$

In particular, if  $n > 0$ ,  $\Pi_{g,n}$  is a free group of rank  $2g + n - 1$ .

The following fact concerning  $\pi_1(X)$  and  $\pi_1^t(X)$  is more or less well-known:

**Lemma.**

*We have the following three surjections (i)–(iii):*

$$\begin{array}{ccc} \pi_1(X) & & \\ \downarrow (i) & & \\ (\Pi_{g,n})^\wedge & \xrightarrow{(ii)} & \pi_1^t(X) \xrightarrow{(iii)} (\Pi_{g,n})^{\sim p'}. \end{array}$$

*Moreover, we have:*

(i) *is an isomorphism*  $\iff n = 0$ ,

(ii) *is an isomorphism*  $\iff (g, n) = (0, 0), (0, 1)$ ,

*and*

(iii) *is an isomorphism*  $\iff \begin{cases} \text{either } (g, n) = (0, 0), (0, 1), (0, 2), \\ \text{or } (g, n) = (1, 0) \text{ and } X \text{ is supersingular.} \end{cases}$

*Proof.* Surjection (i) comes from the definitions of  $\pi_1(X)$  and  $\pi_1^t(X)$  (or the above descriptions of  $\pi_1(X)$  and  $\pi_1^t(X)$  in terms of Galois groups). For surjections (ii) and (iii), see [SGA1].

If  $n = 0$ , i.e.,  $X$  is projective, then the natural surjection  $\pi_1(X) \rightarrow \pi_1^t(X)$  is an isomorphism by definition. On the other hand, if  $n > 0$ , i.e.,  $X$  is affine,  $\pi_1(X)$  is not topologically finitely generated (as its maximal pro- $p$  quotient  $\pi_1(X)^p$  is a free pro- $p$  group of rank  $|k|$ ), while  $\pi_1^t(X)$  is always topologically finitely generated as a quotient of  $(\Pi_{g,n})^\wedge$ . Thus, in this case, (i) cannot be an isomorphism.

Next, if  $(g, n) = (0, 0), (0, 1)$ , then we have  $\Pi_{g,n} = \{1\}$ . Thus, in this case, surjections (ii) and (iii) must be isomorphisms. If  $(g, n) = (0, 2)$  (resp.  $(g, n) = (1, 0)$  and  $X$  is supersingular), then  $\pi_1^t(X)$  coincides with  $(\Pi_{g,n})^{\sim p'} = \widehat{\mathbb{Z}}^{p'}$  (resp.  $\widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'}$ ). Thus, surjection (iii) is then an isomorphism.

On the other hand, assume that (ii) is an isomorphism. Then, the abelianizations  $(\Pi_{g,n})^{\text{ab}}$  and  $\pi_1^t(X)^{\text{ab}}$  are also isomorphic to each other. Observing the pro- $p$  parts, we obtain  $2g + n - 1 + b^{(2)} = \gamma$ , where  $b^{(2)}$  denotes the second Betti number of  $X$  (i.e.,  $b^{(2)} = 1$  for  $n = 0$  and  $b^{(2)} = 0$  for  $n > 0$ ), and  $\gamma$  denotes the  $p$ -rank (or Hasse-Witt invariant) of  $X$ . Since  $\gamma \leq g$ , this equality implies  $(g, n) = (0, 0), (0, 1)$ .

Finally, assume that (iii) is an isomorphism, which implies that  $\pi_1^t(X)$  has a trivial pro- $p$ -Sylow subgroup. If either  $(g, n) = (0, 0), (0, 1), (0, 2)$ , or  $(g, n) = (1, 0)$  and  $X$  is supersingular, nothing remains to be proved. If  $(g, n) = (1, 0)$  and  $X$  is ordinary, then  $\pi_1^t(X) = \pi_1(X)$  is isomorphic to  $\widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'} \times \mathbb{Z}_p$ , hence its pro- $p$ -Sylow subgroup is non-trivial. Finally, if  $(g, n) \neq (0, 0), (0, 1), (0, 2), (1, 0)$ , i.e.,  $X$  is hyperbolic, then there exists a tame covering  $Y \rightarrow X$  such that the genus of  $Y^*$  is not less than 2. By [R1], Corollaire 4.3.2,  $\pi_1(Y^*)$  has a non-trivial pro- $p$ -Sylow subgroup. Since  $\pi_1(Y^*)$  is a subquotient of  $\pi_1^t(X)$ , this implies that  $\pi_1^t(X)$  admits a non-trivial pro- $p$ -Sylow subgroup. This completes the proof.  $\square$

**Example.**

The following are the only cases (for the present) in which we can describe  $\pi_1(X)$  or  $\pi_1^t(X)$  explicitly.

(i) If  $(g, n) = (0, 0)$ , then we have

$$\pi_1(X) = \pi_1^t(X) = \{1\}.$$

(ii) If  $(g, n) = (0, 1)$ , then we have  $\pi_1^t(X) = \{1\}$ .

(iii) If  $(g, n) = (0, 2)$ , then we have

$$\pi_1^t(X) \simeq \widehat{\mathbb{Z}}^{p'}.$$

(iv) If  $(g, n) = (1, 0)$ , then we have

$$\pi_1(X) = \pi_1^t(X) \simeq \begin{cases} \widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'} \times \mathbb{Z}_p, & X: \text{ordinary}, \\ \widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'}, & X: \text{supersingular}. \end{cases}$$

Now, our main question is as follows.

**Main question.**

*Exactly what information on the geometry of  $X$  does  $\pi_1(X)$  (or  $\pi_1^t(X)$ ) carry?*

As for this question, the best situation we can expect is:

**Hope.**

*Assume  $k = \overline{\mathbb{F}}_p$ . (For general  $k$ , see [T3].)*

(i)  $\pi_1(X)$  determines the isomorphism class of  $X$  (as a scheme), unless  $(g, n) = (1, 0)$ .

(ii)  $\pi_1^t(X)$  determines the isomorphism class of  $X$  (as a scheme), unless  $(g, n) = (0, 0), (0, 1), (1, 0)$ .

*Remark (which shows that our problem is rather subtle).*

(i) In characteristic 0, we have  $\pi_1(X) = \pi_1^t(X) \simeq (\Pi_{g,n})^\wedge$ , which carries very little information about  $X$ .

(ii) Let  $\pi_A(X)$  denote the set of isomorphism classes of finite quotient groups of  $\pi_1(X)$ . Then, the Abhyankar conjecture (proved by Raynaud [R2] and Harbater [H1]) asserts that, if  $n > 0$ , we have

$$\pi_A(X) = \{G \mid G^{p'} \text{ is generated by (at most) } 2g + n - 1 \text{ elements.}\} / \simeq.$$

Thus, in this case,  $\pi_A(X)$  carries very little information about  $X$ .

(iii) The geometric Shafarevich conjecture (proved by Harbater [H3] and Pop [P]) asserts that the absolute Galois group  $G_{k(X)}$  is a free profinite group of rank  $|k|$ . Thus,  $G_k(X)$ , which is an extension group of  $\pi_1(X)$ , carries very little information

## §2. Some results (which support our hope).

In this §, we present some results which support our hope in the last §.

### Theorem 0.

- (i) ([T1])  $\pi_1(X)$  determines  $(g, n)$ .
- (ii) ([T2])  $\pi_1^t(X)$  determines  $(g, n)$ , unless  $(g, n) = (0, 0), (0, 1)$ .

### Theorem 1.

Assume  $k = \overline{\mathbb{F}}_p$  and  $g = 0$ .

- (i) ([T1])  $\pi_1(X)$  determines the isomorphism class of  $X$  (as a scheme).
- (ii) ([T2])  $\pi_1^t(X)$  determines the isomorphism class of  $X$  (as a scheme), unless  $n = 0, 1$ .

### Theorem 2 ([PS], [R4], [T4]).

Assume  $k = \overline{\mathbb{F}}_p$ .

- (i)  $\pi_1(X)$  determines the isomorphism class of  $X$  up to finite possibilities, unless  $(g, n) = (1, 0)$ .
- (ii)  $\pi_1^t(X)$  determines the isomorphism class of  $X$  up to finite possibilities, unless  $(g, n) = (1, 0)$ .

One of the main ingredients of the proofs of the above theorems is Raynaud's theory of theta divisors ([R1], cf. [R3], [M]).

## §3. Applications.

In this §, we present two applications of the results in the last §.

### Corollary to Theorem 1(ii) (Tamagawa, unwritten yet).

Let  $L$  be a subfield of  $\overline{\mathbb{Q}}$ , and assume that there exist infinitely many rational primes  $p$ , such that  $I_{\overline{p}} \cap G_L \not\subset I_{\overline{p}}^w$  holds for some prime  $\overline{p}$  of  $\overline{\mathbb{Q}}$  above  $p$ . (Here,  $I_{\overline{p}}$  (resp.  $I_{\overline{p}}^w$ ) denotes the inertia subgroup (resp. the pro- $p$ -Sylow subgroup of the inertia subgroup) at  $\overline{p}$ .) Let  $X$  be a smooth, hyperbolic curve over  $L$ , and assume that the smooth compactification of  $X$  is of genus 0. Then, the outer Galois representation  $\rho : G_L \rightarrow \text{Out}(\pi_1(X_{\overline{\mathbb{Q}}}))$  determines the isomorphism class of  $X$  as an  $L$ -scheme.

### Corollary to Theorem 2(ii) ([T4]).

Let  $F$  be a function field of one variable over a finite field of characteristic  $p$ . Let  $X$  be a smooth, hyperbolic curve over  $F$ , and assume that  $X$  is non-isotrivial. Let  $\rho^t$  denote the outer Galois representation  $G_F \rightarrow \text{Out}(\pi_1^t(X_{\overline{F}}))$ . Then, for all primes  $P$  of  $F$ , we have

$$\text{Ker}(\rho^t) \cap D_P \subset I_P^w.$$

(Here,  $D_P$  (resp.  $I_P^w$ ) denotes the decomposition subgroup (resp. the pro- $p$ -Sylow subgroup of the inertia subgroup) at  $P$ , defined up to conjugacy.)

### Remark.

Let  $F$  be an arbitrary field of characteristic  $> 0$ , and  $X$  a smooth, affine curve over  $F$ . Then, the outer Galois representation  $\rho : G_F \rightarrow \text{Out}(\pi_1(X_{\overline{F}}))$  is injective (see

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